1996

Solution | *Scoring Scale*

Points

(a) volume $= 2\pi \int_0^\infty x e^{-x^2}\, dx$

$= 2\pi \lim_{b\to\infty} \int_0^b x e^{-x^2}\, dx = 2\pi \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^b$

$= 2\pi \lim_{b\to\infty} \left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^0\right)$

$= 2\pi \left(\frac{1}{2}\right) = \pi$, or

volume $= \pi \int_0^1 \left(\sqrt{(-\ln y)}\right)^2 dy$

$= -\pi \lim_{a\to 0+} \int_a^1 (\ln y)\, dy = \pi$

5
- 2 : writes V as integral
  - 1 : integrand (includes constant)
  - 1 : limits
- 2 : expresses improper integral in terms of a proper integral
- 1 : answer from an improper integral

max 4/5: misuses infinity, 2 - 1 - 1.

(b) maximum:

$A(w) = we^{-w^2}$,

$A'(w) = e^{-w^2} - 2w^2e^{-w^2} = e^{-w^2}(1-2w^2)$.

$A'(w) > 0$ when $w < 1/\sqrt{2}$,

$A'(w) = 0$ when $w = 1/\sqrt{2}$,

$A'(w) < 0$ when $w > 1/\sqrt{2}$.

Therefore, max occurs when $w = 1/\sqrt{2}$.

inflection:

$h(x) = e^{-x^2}$, $h'(x) = -2xe^{-x^2}$,

$h''(x) = -2e^{-x^2} - 2x(-2x)e^{-x^2}$

$= 2e^{-x^2}(-1 + 2x^2)$.

$h''(x) < 0$ when $x < 1/\sqrt{2}$,

$h''(x) = 0$ when $x = 1/\sqrt{2}$,

$h''(x) > 0$ when $x > 1/\sqrt{2}$.

Therefore, inflection point when $x = 1/\sqrt{2}$.

Therefore, the maximum value of $A(w)$ and the inflection point of $h(x)$ occur when x and w are $1/\sqrt{2}$.

4
- 1 : $A(w)$
- 1 : sets $A'(w) = 0$
- 1 : sets $h''(x) = 0$
- 1 : indicates that the equations have the same solution

1996

Solution | *Scoring Scale*

Points

(a) $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{(n+1)!}$

$f'(0) = a_1 = \dfrac{1}{2}$

$f^{(17)}(0) = 17!\, a_{17} = 17!\left(\dfrac{1}{18!}\right) = \dfrac{1}{18}$

2 $\begin{cases} 1: f'(0) \\ 1: f^{(17)}(0) \end{cases}$

(b) $$\lim_{n\to\infty} \frac{\left|\dfrac{x^{n+1}}{(n+2)!}\right|}{\left|\dfrac{x^n}{(n+1)!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$$

converges for all x, by ratio test

2 $\begin{cases} 1: \text{sets up ratio test and finds the limit} \\ 1: \text{applies ratio test to get answer} \end{cases}$

(c) $g(x) = xf(x)$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

2 $\begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \\ <-1>: \text{not an infinite series} \end{cases}$

(d) $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$e^x - 1 = g(x) = xf(x)$

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

3 $\begin{cases} 1: g(x) = e^x - 1 \\ 1: f(x) = \dfrac{e^x - 1}{x} \\ \quad (\text{if } g(x) \text{ is incorrect must include } e^x) \\ 1: \text{correctly considers } x = 0 \text{ case} \end{cases}$

1996

Solution | *Scoring Scale*

Points

(a) $S(t) = Ce^{kt}$

$S(0) = 6 \implies C = 6$

$S(5) = 12 \implies 12 = 6e^{5k}$

$2 = e^{5k}$

$k = \dfrac{\ln 2}{5}$ (0.138 or 0.139)

$$3 \begin{cases} 1: C = 6 \\ 1: 12 = 6e^{5k} \\ 1: k = \dfrac{\ln 2}{5} \end{cases}$$

(b) average rate $= \dfrac{1}{13-3}\displaystyle\int_3^{13} 6e^{\frac{\ln 2}{5}t}\,dt$

$= \dfrac{3}{\ln 2}\left[e^{2.6\ln 2} - e^{0.6\ln 2}\right]$ billion gal/yr

(19.680 billion gal/yr)

$$3 \begin{cases} 1: \text{uses } [3,13] \text{ and divides by } 13-3 \\ 1: \text{integrand} \\ 1: \text{answer with units} \\ \quad 0/1 \text{ if not } \dfrac{1}{b-a}\displaystyle\int_a^b S(t)\,dt \end{cases}$$

(c) $\displaystyle\int_5^7 S(t)\,dt$

$\doteq \dfrac{1}{4}[S(5) + 2S(5.5) + 2S(6) + 2S(6.5) + S(7)]$

$$1 \begin{cases} \text{Trapezoidal rule with } S, \\ n = 4, \text{ interval } [5,7] \end{cases}$$

(d) This gives the total consumption, in billions of gallons, during the years 1985 and 1986.

$$2 \begin{cases} 1: \text{total consumption in a time period} \\ 1: \begin{cases} \text{correct time period} \\ \text{liquid measure} \end{cases} \end{cases}$$

0/2 for "rate of consumption"

Board Note
AB-3, BC-3

Part(b)

Student solves $\dfrac{S(13) - S(3)}{10} = S'(t)$ for t_0.

$S(t_0) = 19.680$ billions gal/yr

3/3 if correct.

Otherwise, read by standard where max 1 - 0 - 0.

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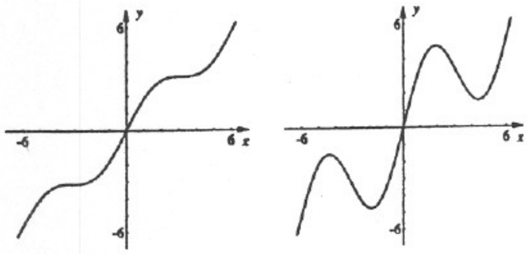
Solution | *Scoring Scale*

Points

(a)



2 $\begin{cases} 1: \text{graph of } y = x + \sin x \\ \quad \text{(increasing, through origin, cd - cu - cd - cu)} \\ 1: \text{graph of } y = x + 3\sin x \\ \quad \text{(inc - dec - inc - dec - inc, through origin, cd - cu - cd - cu)} \end{cases}$

(b) $y' = 1 = 1 + b\cos x$

$b\cos x = 0$

$\cos x = 0$

$y = x + b = x + b\sin x$

$b = b\sin x$

$1 = \sin x$

$x = -\dfrac{3\pi}{2}$ or $\dfrac{\pi}{2}$

4 $\begin{cases} 1: \text{derivatives of } x + b \text{ and } x + b\sin x \\ 1: \text{sets derivatives equal} \\ 1: \text{sets } y\text{-values equal} \\ 1: \text{answer from merged information} \end{cases}$

max: 0 - 1 - 1 - 0 if specific value(s) for b

max: 0 - 0 - 1 - 0 if no derivative

(c) No, because $f'(x) = 1$ (or $f'(x) \neq 0$) at x-coordinates of points of tangency.

1: answer with reason

(d) $f''(x) = -b\sin x = 0$

$\sin x = 0$

$f(x) = x + b \cdot 0 = x$
at x-coordinates of any inflection points

2 $\begin{cases} 1: \text{sets second derivative equal to } 0 \\ 1: \text{shows } f(x) = x \end{cases}$

0/2 if specific value(s) for b

1996

Solution

Scoring Scale

(a) $V = \pi \int_0^h \frac{25}{3}\sqrt{y}\,dy$

$$\frac{dV}{dt} = \frac{25\pi}{3}\sqrt{h}\,\frac{dh}{dt}$$

At $h = 4$, $8 = \frac{25\pi}{3}\sqrt{4}\,\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{12}{25\pi} \text{ ft/min}$$

Points

5
- 1 : volume as an integral using h
- 2 : finding $\frac{dV}{dt}$:
  - 1 : $\frac{dV}{dh}$
  - 1 : $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$
- 1 : $\frac{dV}{dt} = 8$
- 1 : answer with units

if V is linear, max 2/5 (0 - 0 - 1 - 1 - 0)
if V is constant, max 1/5 (0 - 0 - 0 - 1 - 0)

(b) $W = 50 \int_0^6 (9 - y)\left(\frac{25\pi}{3}\sqrt{y}\right) dy$

$$W = 50\left(\frac{25\pi}{3}\right)\int_0^6 \left(9y^{\frac{1}{2}} - y^{\frac{3}{2}}\right) dy$$

$$W = 50\left(\frac{25\pi}{3}\right)\left(\frac{2}{3}\cdot 9y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_0^6$$

$W = 69{,}257.691$ ft-lbs

to the nearest foot-pound: 69,258 ft-lbs

4
- 1 : limits
- 2 : integrand :
  - 1 : force
  - 1 : distance
- 1 : answer

  0/1 if no integrand points

1996

Solution

(a) $\tan\theta_1 = \frac{1}{1} \implies \theta_1 = \frac{\pi}{4}$ or 0.785

$\tan\theta_2 = \frac{25}{5} \implies \theta_2 = \tan^{-1} 5$ or 1.373

Therefore, $\frac{\pi}{4} \le \theta \le \tan^{-1} 5$

(b) $\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$

Therefore, $x = \tan\theta \qquad y = x^2 = \tan^2\theta$

(c) $\frac{d\theta}{dt} = 2\pi$

$\frac{dx}{dt} = \sec^2\theta \frac{d\theta}{dt}; \quad \frac{dy}{dt} = 2\tan\theta\sec^2\theta\frac{d\theta}{dt}$

at $(3,9)$: $\frac{dx}{dt} = 10 \cdot 2\pi = 20\pi$

$\frac{dy}{dt} = 2 \cdot 3 \cdot 10 \cdot 2\pi = 120\pi$

speed $= \sqrt{(20\pi)^2 + (120\pi)^2}$

$= 20\pi\sqrt{37}$ or 382.191

Scoring Scale

Points

(a) $2 \begin{cases} 1 : \theta_1 \\ 1 : \theta_2 \end{cases}$

$<-1>$: no interval given

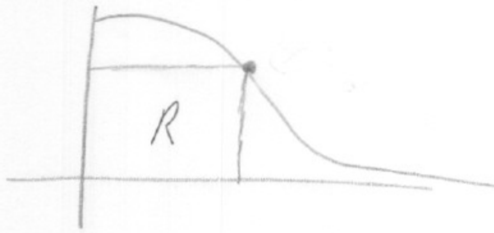
1/2 : complete solution in degrees

(b) $2 \begin{cases} 1 : x = \tan\theta \\ 1 : y = \tan^2\theta \end{cases}$

(c) $5 \begin{cases} 1 : \frac{d\theta}{dt} = 2\pi \\ 1 : \frac{dx}{dt} \text{ using Chain Rule} \\ 1 : \frac{dy}{dt} \text{ using Chain Rule} \\ 1 : \text{student uses magnitude of velocity} \\ 1 : \text{answer} \end{cases}$

1996 BC #1

a.



$$V = 2\pi \int_0^\infty x e^{-x^2}\,dx$$

$$= 2\pi\left(-\tfrac{1}{2}e^{-x^2}\right)\Big]_0^\infty$$

$$= -\pi e^{-x^2}\Big]_0^\infty$$

$$= -\pi \frac{1}{e^{\infty^2}} + \pi \cancel{e^{0}}^{\,1}$$

$$= 0 + \pi = \pi$$

b. $\text{Area} = w e^{-w^2}$

$$A' = w e^{-w^2}(-2w) + e^{-w^2}(1)$$

$$= e^{-w^2}(-2w^2+1)$$

$e^{-w^2} = 0$ never

$$-2w^2+1=0$$
$$-2w^2=-1$$
$$w^2=\tfrac{1}{2}$$
$$w=\sqrt{\tfrac{1}{2}}$$

	$\sqrt{\frac{1}{2}}$
$-2w^2+1$	+ 0 −
A'	+ 0 −

^ MAX

b. Continued

$$h = e^{-x^2}$$

$$h' = e^{-x^2}(-2x)$$

$$h'' = e^{-x^2}(-2) + (-2x)e^{-x^2}(-2x)$$

$$= -2e^{-x^2}(1-2x^2)$$

	$\sqrt{2}$
$1-2x^2$	+ 0 −
h''	+ 0 −

↑ pt of inflection

∴ Max of w and point of inflection for x are both $\sqrt{\frac{1}{2}}$

1996 BC #2

a. $f'(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \cdots \frac{nx^{n-1}}{(n+1)!}$

$f'(0) = \frac{1}{2} + \frac{0}{3!} + \frac{0}{4!} + \cdots$

$= \frac{1}{2}$

To find $f^{17}(0)$ explore a few more terms

$f''(x) = \frac{2}{3!} + \frac{3 \cdot 2x}{4!} + \frac{4 \cdot 3x^2}{5!} + \cdots$

$f''(0) = \frac{2}{6} + \frac{6(0)}{4!} + \frac{12(0)}{5!} + \cdots = \frac{1}{3}$

$f'''(x) = \frac{6}{4!} + \frac{24}{5!}x + \cdots$

$f'''(0) = \frac{1}{4}$ ← always one more than derivative

$\therefore f^{17}(0) = \frac{1}{18}$

b. Ratio Test

$$\lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{((n+1)+1)!}}{\frac{x^n}{(n+1)!}} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^n \cdot x \, (n+1)!}{(n+1)!(n+2)x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x}{n+2} \right| = 0$$

$\therefore$ It converges for all real numbers

1996 BC #2

c. $g(x) = xf(x)$

$$= x\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right]$$

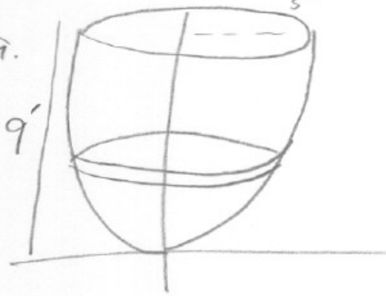
$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \frac{x^{n+1}}{(n+1)!}$$

d. $g(x) = e^x - 1$

$$f(x) = \frac{g(x)}{x} = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

1996 BC 5

a.



Use horizontal Disks

$$y = \frac{9}{625}x^4$$

$$\frac{625}{9}y = x^4$$

$$\frac{5}{\sqrt{3}}y^{\frac{1}{4}} = x$$

$$V = \pi\int_0^h x^2\,dy$$

$$= \pi\int_0^h \left(\frac{5}{\sqrt{3}}y^{\frac{1}{4}}\right)^2 dy$$

$$= \pi\int_0^h \frac{25}{3}\sqrt{y}\,dy$$

$$= \pi\frac{25}{3}\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big]_0^h$$

$$= \pi\frac{25}{3}\cdot\frac{2}{3}h^{\frac{3}{2}}$$

$$V = \frac{50\pi}{9}h^{\frac{3}{2}}$$

b) See Solution Book

$$\frac{dV}{dt} = \frac{50\pi}{9}\frac{3}{2}h^{\frac{1}{2}}\frac{dh}{dt}$$

$$= \frac{25}{3}\sqrt{h}\,\frac{dh}{dt}$$

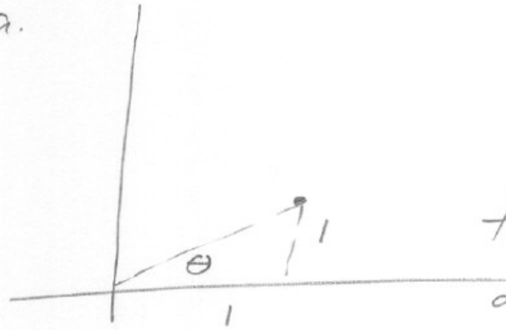
let $h=4$

$$8 = \frac{25}{3}\sqrt{4}\,\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{12}{25\pi} \approx .153 \text{ ft/min}$$

1996 BC 6

a.



$\tan\theta = \frac{1}{1}$

$\arctan \frac{1}{1} = \theta$

$\therefore \quad \arctan \frac{1}{1} \le \theta \le \arctan \frac{25}{5}$

$\arctan 1 \le \theta \le \arctan 5$

$\frac{\pi}{4} \le \theta \le 1.373$

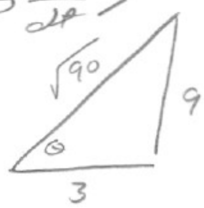
b. $\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$

$\therefore \quad x = \tan\theta$

$y = x^2 = (\tan\theta)^2$

c. Position Vector $\langle \tan\theta, \tan^2\theta \rangle$

Velocity Vector $= \langle \sec^2\theta \frac{d\theta}{dt}, 2\tan\theta \sec^2\theta \frac{d\theta}{dt} \rangle$



Point (3, 9)

$\frac{d\theta}{dt} = 2\pi$ units per min

$\tan\theta = \frac{9}{3} = 3 \quad \sec\theta = \frac{\sqrt{90}}{3} = \sqrt{10}$

$|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left[\sec^2\theta \frac{d\theta}{dt}\right]^2 + \left[2\tan\theta \sec^2\theta \frac{d\theta}{dt}\right]^2}$

$= \sqrt{\sec^4\theta \left(\frac{d\theta}{dt}\right)^2 + 4\tan^2\theta \sec^4\theta \left(\frac{d\theta}{dt}\right)^2}$

$= \sqrt{(\sqrt{10})^4 (2\pi)^2 + 4(3)^2 (\sqrt{10})^4 (2\pi)^2}$

$= 20\pi\sqrt{37} = 382.191$ units/min